Non-Markovian noise

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The properties of non-Markovian noises with exponentially correlated memory are discussed. Considered are dichotomic noise, white shot noise, Gaussian white noise, and Gaussian colored noise. The stationary correlation functions of the non-Markovian versions of these noises are given by linear combinations of two or three exponential functions (colored noises) or of the δ function and exponential function (white noises). The non-Markovian white noises are well defined only when the kernel of the non-Markovian master equation contains a nonzero admixture of a Markovian term. Approximate equations governing the probability densities for processes driven by such non-Markovian noises are derived, including non-Markovian versions of the Fokker-Planck equation and the telegrapher's equation. As an example, it is shown how the non-Markovian nature changes the behavior of the driven linear process.

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I. INTRODUCTION

The physical, chemical, biological, etc., systems driven by various types of noises, stochastic fluctuations, and random processes have been a subject of theoretical and experimental investigations for many years [1]. For theoretical studies a model of noise is needed. The simplest models are the uncorrelated (white) noises, especially the Gaussian white noise (GWN), generated by the Wiener stochastic process. For many years GWN has been used almost universally, especially in applications, mainly because the stochastic flows driven by GWN can be elegantly described by appropriate Fokker-Planck equation. The so-called white shot noise (WSN) generated by another elementary stochastic process, viz. the Poisson one, is also used, though rather rarely [2,3]. White noises, especially the GWN, have some unphysical properties and their use requires some care (cf. [4–6]). Thus, in the last two decades the attention turned to more physical colored noises with finite correlation times. Of these, most frequently used is the Gaussian colored noise (GCN), being the natural "extension" of GWN (GCN is generated by the Ornstein-Uhlenbeck process, i.e., the relaxation process driven by Wiener process). However, the use of GCN in applications causes some difficulties: for stochastic flows driven by GCN the corresponding Fokker-Planck equation is but an approximation [6,7]. Hence, more and more attention has been paid in the last decade to another approximation of real random disturbances, the so-called dichotomic noise (DN), being the realization of two-state Poisson process [2,8–12]. Its main assets are (i) DN is colored, (ii) application of DN results in relatively simple calculations [13], (iii) well-defined limiting procedures lead from DN to both GWN and WSN [2]. Some other types of noise were discussed in literature occasionally; of these, worth mentioning seem to be quantum noises, both from the fundamental point of view [14], and in application to the theory of lasers [15], hyperbolic sine model [16], quadratic noise [17], and composite noises [18].

One common feature of all the work mentioned above is that the noises used as driving stochastic processes are (to the best of author's knowledge) almost without exception Markovian [19]. Only very recently a few papers have been published which deal explicitly with non-Markovian driving: Pawula *et al.* used DN's with nonexponential distributions of switching rates as driving processes in calculation of mean first passage times [20]. Besides, papers discussing master equations with time-dependent transition rates [21] and "non-Markovian Fokker-Planck equation" [22] (i.e., Fokker-Planck equation in energy variable) have been published during last year. Periodically driven linear non-Markovian systems are mentioned by Jung [23].

Markovianity assumption seems to be also an idealization of both real internal fluctuations and real random external disturbances. Fundamental derivations of Langevin-type equations from basic equations of mechanics suggest that the proper approximations for intrinsic fluctuations (which are related to the averaged-out fast processes) should be either non-Markovian or nonstationary Markovian and that the Markovian approximations may be hard to justify in some cases [28]. Also, every real process has some intrinsic built-in delay or inertia [24], which in the stochastic theory language means the presence of memory, i.e., non-Markovianity of external (parametric) noise. Therefore, one may expect that at least in some situations the non-Markovian models of driving noises would be better approximations than the Markovian ones.

In this paper, non-Markovian variants of DN, WSN, GWN, and GCN are being proposed. We shall define the non-Markovian DN by general master equation containing explicit memory, and not by specification of probability distribution for switching times. White non-Markovian noises will be defined as appropriate limits

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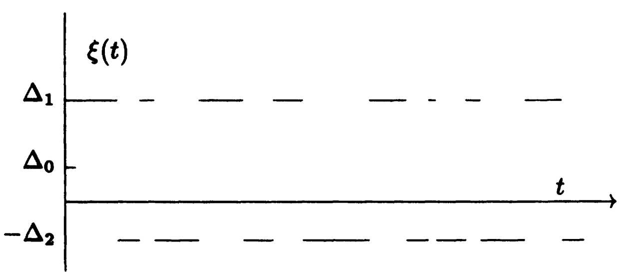
of non-Markovian DN, and non-Markovian GCN by the Ornstein-Uhlenbeck process driven by non-Markovian GWN.

The rest of the paper is organized as follows. Section II A contains elementary definition of the non-Markovian dichotomic noise and the derivation of its basic properties: time dependence of one- and two-point probability distributions, two-point correlation function, and correlation times; in Sec. II B the same quantities are derived for non-Markovian white noises. Section III deals with the higher-order distributions, necessary for the complete definition of a non-Markovian process. General stochastic flows driven by the noises considered in Sec. II are discussed in Sec. IV; there the equations governing the probability densities for driven process and their stationary solutions are derived. In Sec. V the properties of a linear process driven by non-Markovian noises are calculated; when the driving process is the non-Markovian GWN, the driven process becomes the non-Markovian Ornstein-Uhlenbeck one, i.e., the non-Markovian version of the colored Gaussian noise. The last section contains some concluding remarks.

II. NON-MARKOVIAN NOISES

A. Dichotomic noise

Consider the asymmetric random two-state process (random telegraph signal) $\xi(t)$ with zero mean:



called the dichotomic noise (DN):

$$\xi(t) \in \{\Delta_1, -\Delta_2\}\,, \quad \xi^2(t) = \Delta^2 + \Delta_0 \xi(t)\,, \quad \langle \xi(t) \rangle = 0\,, \tag{2.1}$$

with $\Delta^2 = \Delta_1 \Delta_2$, $\Delta_0 = \Delta_1 - \Delta_2$.

Let λ_1 and λ_2 be the probabilities of switching (per unit time) between states Δ_1 and Δ_2, so that $\tau_i = 1/\lambda_i$ are the mean sojourn times in these states. $\langle \xi(t) \rangle = 0$ means, thus, that

$$\Delta_1/\lambda_1 = \Delta_2/\lambda_2 = w_0\,, \tag{2.2}$$

and that the unconditional probability $P_1(\xi, t)$ of finding the process in the state ξ at time $t, t+dt$ is always

$$P_1(\xi,t) = P_{st}(\xi) = \Lambda^{-1}[\lambda_2 \delta_{\Delta_1,\xi} + \lambda_1 \delta_{-\Delta_2,\xi}] = [\tau_1 \delta_{\Delta_1,\xi} + \tau_2 \delta_{-\Delta_2,\xi}]/(\tau_1+\tau_2) \tag{2.3}$$

($\Lambda = \lambda_1 + \lambda_2$) for all $t \geq t_0$, i.e., that the dichotomic process $\xi(t)$ with zero mean is stationary.

Let $P(\xi, t|x, t_0) = P_{1|1}(\xi, t|x, t_0)$ denote the conditional probability that the process is in the state ξ at time interval $(t, t+dt)$ given that it was in the state x at time t_0. We assume that the master equation governing the behavior of $P(\xi, t|x, t_0)$ contains the non-Markovian ("memory") term and reads

$$\dot{P}(\Delta_1, t|x, t_0) = -\int_{t_0}^{t} dt' K(t-t')\big[\lambda_1 P(\Delta_1, t'|x, t_0) - \lambda_2 P(-\Delta_2, t'|x, t_0)\big]\,,$$

$$\dot{P}(-\Delta_2, t|x, t_0) = \int_{t_0}^{t} dt' K(t-t')\big[\lambda_1 P(\Delta_1, t'|x, t_0) - \lambda_2 P(-\Delta_2, t'|x, t_0)\big]\,, \tag{2.4}$$

or (we follow here Ref. [8])

$$\dot{\phi}(t) = -\Lambda \int_{t_0}^{t} dt' K(t-t')\phi(t')\,, \tag{2.5}$$

where

$$\phi(t) = \lambda_1 P(\Delta_1, t|x, t_0) - \lambda_2 P(-\Delta_2, t|x, t_0) = \Lambda P(\Delta_1, t|x, t_0) - \lambda_2\,. \tag{2.6}$$

The general form (2.4) of the master equation can be derived exactly and directly from the basic quantum-mechanical equations of motion [25]. It is to be noted that from this point of view the form (2.4) containing explicit "memory" is fully equivalent, by identity transformations, to the "memoryless" form containing instead time-dependent coefficients, both for exact generalized master equations [26], and for exact generalized Langevin equations [27]. This suggests that the choice of the form (2.4) or of the equivalent form without time integration but with time-dependent transition matrix is largely the matter of convenience.

Virtually nothing is known about the general form of the memory kernel $K(t-t')$. There are neither experimental nor theoretical data in this respect. The general theory mentioned above suggests that the exact form of $K(t-t')$ will be system dependent—K is given by appropriate averages of fast processes eliminated from the description [25–29]. Therefore the probably safest and most sensible, both from physical and from practical point of view is to assume that the memory kernel K has the form

$$K(t-t') = \gamma_0 \delta(t-t') + \gamma_1 e^{-\nu(t-t')}\,, \tag{2.7}$$

which contains both Markovian and non-Markovian contributions. This allows for the continuous change from

Markovian to non-Markovian dynamics and enables identification of terms of Markovian and non-Markovian origin. Moreover, some limiting procedures are well defined only when the nonzero contribution of Markovian dynamics is present. For $\gamma_1 = 0$ (Markovian case) these equations become identical with those in Ref. [8]. The exponential damping of the memory kernel is the most popular simplification, with several obvious advantages. Besides, exponential relaxation is frequently found in many physical, chemical, etc., processes, at least at final stages of evolution. Therefore, the form (2.4) with memory kernel (2.7) seems to be general enough from the physical point of view, allowing at the same time for exact solutions of Eqs. (2.4).

The solution to these equations (valid for $t > t_0$) reads

$$\begin{aligned} \phi(t) &= \psi(t-t_0)\phi_0(x)\,, \\ P(\xi,t|x,t_0) &= P_{st}(\xi) + \Lambda^{-1}\psi(t-t_0)\psi_0(\xi)\phi_0(x)\,, \end{aligned} \tag{2.8}$$

with

$$\begin{aligned} \phi_0(x) &= \phi(t_0) = \lambda_1\delta_{\Delta_1,x} - \lambda_2\delta_{-\Delta_2,x}\,, \\ \psi_0(\xi) &= \delta_{\Delta_1,\xi} - \delta_{-\Delta_2,\xi}\,, \\ \psi(t) &= \Gamma^{-1}\left[(\theta_1-\nu)e^{-\theta_1 t} - (\theta_2-\nu)e^{-\theta_2 t}\right], \\ \theta_{1,2} &= \tfrac{1}{2}\left(\nu + \gamma_0\Lambda \pm \Gamma\right), \\ \Gamma &= \sqrt{(\gamma_0\Lambda-\nu)^2 - 4\gamma_1\Lambda}\,, \end{aligned} \tag{2.9}$$

where the identity $P(x',t_0|x,t_0) = \delta_{x',x}$ was used. Note that for Markovian process, $\gamma_1 = 0$, the time function ψ reads

$$\psi(t) = e^{-\gamma_0\Lambda t}\,. \tag{2.10}$$

Therefore

$$\lim_{t-t_0\to\infty} P(\xi,t|x,t_0) = P_{st}(\xi)\,. \tag{2.11}$$

The considered non-Markovian process is irreversible and its stationary distributions do not remember initial state. Moreover, the stationary distributions are the same as for the Markovian process. Note that these formulas are valid only for $t > t_0$.

Equations (2.8) and (2.3) together with the Bayes rule [8] give the two-point probability and the two-point (stationary) correlation function:

$$\begin{aligned} P_2(\xi,t;\xi',t') &= P(\xi,t|\xi',t')P_1(\xi',t') \\ &= P_{st}(\xi)P_{st}(\xi') \\ &\quad +(\lambda_1\lambda_2/\Lambda^2)\psi_0(\xi)\psi_0(\xi')\psi(t-t')\,, \end{aligned} \tag{2.12}$$

$$\begin{aligned} C(t,t') = \langle\xi(t)\xi(t')\rangle &= \sum_{\xi,\xi'}\xi\xi' P_2(\xi,t;\xi',t') \\ &= \Delta^2\psi(|t-t'|)\,. \end{aligned} \tag{2.13}$$

Thus, for non-Markovian process the stationary correlation function ceases to be simply exponential. It becomes the combination of two exponentials, and moreover, for some combinations of parameters γ_0, γ_1, Λ, and ν, the correlations may become damped oscillatory.

The formula for variance, $\langle\xi^2(t)\rangle = \Delta^2$ results both from (2.13) and directly from the definition of the DN, $\xi^2(t) = \Delta^2 + \Delta_0\xi(t)$, $\langle\xi(t)\rangle = 0$.

These formulas give main characteristics of non-Markovian dichotomic noise. Here γ_0 and γ_1 describe the relative contributions of Markovian and non-Markovian parts, and ν the rate of damping of the non-Markovian memory. For $\gamma_1 = 0$, $\gamma_0 = 1$ we recover the formulas for Markovian DN [8]. From the physical meaning of these parameters we have that $\nu > 0$ and $\Lambda > 0$; when one of $\gamma_i = 0$, the other must be positive, otherwise the process $\xi(t)$ would become divergent. When both Markovian and non-Markovian kinetics are present (both $\gamma_i \neq 0$), γ_i can be either positive or negative, with some limitations. Namely, the proper convergence condition is that both $\theta_i > 0$. Therefore $\nu+\gamma_0\Lambda > 0$, and either $4\gamma_1\Lambda > (\nu-\gamma_0\Lambda)$ or $4\gamma_1\Lambda < (\nu-\gamma_0\Lambda)$ and $\gamma_0+\gamma_1 > 0$.

B. White noises

The white noises (WN) can be obtained from DN as the following limits [2]:

$$\lambda_1 \to \infty\,, \Delta_1 \to \infty\,, \Delta_1/\lambda_1 = \Delta_2/\lambda_2 = w_0\,, \tag{2.14}$$

for WSN, and for GWN

$$\lambda_1 = \lambda_2 = \lambda \to \infty\,,\quad \Delta_1 = \Delta_2 = \Delta \to \infty\,,$$

$$\Delta^2/2\lambda = D_0 = \lambda_2^2 w_0\,. \tag{2.15}$$

Moreover, in taking these limits it is implicitly assumed that

$$\lim_{\lambda\to\infty} \lambda e^{-\lambda t} = \delta(t)\,. \tag{2.16}$$

In both these WN limits the probability densities are rather uninteresting. More interesting is the behavior of the stationary correlation function (2.13). Application of the above limits gives (cf. Appendix A):

$$C(t,t') = \frac{D_0}{\gamma_0}\left[\delta(t-t') - \mu e^{-\theta_0|t-t'|}\right], \tag{2.17}$$

where $\Lambda_0 = \lambda_1$ for WSN, $\Lambda_0 = 2\lambda$ for GWN, and $\theta_0 = \nu + \mu$, $\mu = \gamma_1/\gamma_0$.

The above non-Markovian WN's are characterized by nonzero correlation time $1/\theta_0$. Therefore one may ask whether they are really "white." This question will be addressed in more detail below, in Sec. VI. Here let us assume, for the time being, that we shall be using the term "WN" for convenience.

The WN limits are obtained under the assumption that $\gamma_0 \neq 0$, i.e., that there is some nonzero admixture of Markovian process. For purely non-Markovian process, $\gamma_0 = 0$, these limits cease to be well defined. When $\gamma_0 \neq 0$, the expression under square root in Γ, Eq. (2.9), is dominated by the term $+\gamma_0^2\Lambda^2$ in the limit $\Lambda \to \infty$ and both inverse correlation times θ_i become real, independently of other parameters (one of θ's goes to infinity, the second one remains bounded—cf. Appendix A). However, when $\gamma_0 = 0$, the expression under the square root is dominated by the term $-\gamma_1\Lambda$, Γ becomes imaginary for all values of other parameters (for purely non-Markovian process $\gamma_1 > 0$), and the time dependence of probability densities and of correlation function becomes damped oscillatory, with oscillation frequency going to infinity (cf. Appendix A). In ordinary sense such limits do not exist. This means that the WN limits of DN are well defined only for mixed process. The determination of the WN limits for purely non-Markovian process ($\gamma_0 = 0$) depends on the interpretation of oscillating functions in the infinite frequency limits.

III. HIGHER-ORDER QUANTITIES

A. Distributions

In contrast to Markovian processes, one- and two-point distributions and correlation functions are insufficient for the complete definition of a non-Markovian process. Some further assumptions are needed. The choice of such prescription will complete the definition of the considered non-Markovian process, given in the preceding section.

It is easy to check that the two-point probability (2.12) is the solution of the master equation (2.4) with the initial condition for the equal-time probability:

$$P_2(\xi,t;\xi',t) = P_{st}(\xi)\delta_{\xi,\xi'} . \tag{3.1}$$

This relation results from (2.3) and from the basic definition of n-point probability:

$$P_n(\xi_1,t_1;\dots;\xi_n,t_n) = \langle \delta(\xi(t_1),\xi_1)\cdots\delta(\xi(t_n),\xi_n)\rangle , \tag{3.2}$$

which implies that

$$P_{n+1}(\xi,t_1;\xi_1,t_1,\dots,\xi_n,t_n) = \delta_{\xi,\xi_1} P_n(\xi_1,t_1,\dots,\xi_n,t_n) . \tag{3.3}$$

Here, the averaging is over all realizations of the process $\xi(t)$ between some initial time $t_0 \le \min(t_1,\dots,t_n)$ and $t_{\max} = \max(t_1,\dots,t_n)$, and $\delta(m,n)$ is the convenient notation for the Kronecker symbol $\delta_{m,n}$.

Equation (3.3) together with the Bayes rule implies in turn that

$$P_{1|n}(\xi,t_1|\xi_1,t_1,\dots,\xi_n,t_n) = \delta_{\xi,\xi_1} . \tag{3.4}$$

Therefore, the most natural assumption for the considered non-Markovian DN seems to be that every higher-order probability $P_{n+1}(\xi,t;\xi_1,t_1;\dots;\xi_n,t_n)$, as the function of $t > \forall t_i$, $i = 1,\dots,n$, fulfills the same master equation (2.4) with the same transition probabilities λ_1, λ_2. This leads to the solution:

$$P_{n+1}(\xi,t;\xi_1,t_1;\dots;\xi_n,t_n) = [P_{st}(\xi) + \Lambda^{-1}\psi_0(\xi)\psi(t-t_1)\phi_0(\xi_1)]P_n(\xi_1,t_1;\dots;\xi_n,t_n) , \tag{3.5}$$

which together with Eqs. (2.8)–(2.11) determines by recurrence all probability distributions of the considered non-Markovian dichotomic process.

In the view of the basic definition (3.2), the above assumption may be understood as the consequence of a stronger assumption, viz. that the master equation (2.4) is fulfilled by the one-point probability $\delta(\xi(t),\xi_i)$ for every realization of the stochastic process $\xi(t)$ separately, i.e., that the averaging over all possible realizations does not introduce any essentially new features [30]. In other words, there are underlying assumptions (i) that the averaging process erases only the unimportant details of individual realizations, leaving intact the main characteristics, (ii) that every realization is characterized by the same sojourn times λ_i, and (iii) that different realizations are uncorrelated. This seems to be sensible from the physical point of view, though the realization of these conditions depends on the physical conditions in which occur both the considered process, and the measurement (averaging) process. Nevertheless, the assumption (3.5) is weaker than the assumptions discussed above.

One of the consequences of the above assumptions are the formulas (3.11)–(3.14) below describing the higher correlation functions of the considered non-Markovian noise. The calculations illustrating these consequences are presented in the Appendix B.

By the Bayes rule the conditional probability has the form

$$P_{1|n}(\xi,t|\xi_1,t_1;\dots;\xi_n,t_n) = P_{st}(\xi) + \Lambda^{-1}\psi_0(\xi)\psi(t-t_1)\phi_0(\xi_1) . \tag{3.6}$$

Again, it is easy to check that the same results can be obtained from master equations (2.4) written for the conditional probabilities $P_{1|n}(\xi,t|\xi_1,t_1;\dots;\xi_n,t_n)$ with the initial condition (3.4).

These distributions have the following property: because

$$\lim_{t\to\infty}\psi(t-t_i) = 0 \text{ for } t_i < \infty , \tag{3.7}$$

then

$$\lim_{t\to\infty} P_{n+1}(\xi,t;\xi_1,t_1;\dots;\xi_n,t_n) = P_{st}(\xi)P_n(\xi_1,t_1;\dots;\xi_n,t_n) \tag{3.8}$$

and

$$\lim_{t\to\infty} P_{1|n}(\xi,t|\xi_1,t_1;\dots;\xi_n,t_n) = P_{st}(\xi) \tag{3.9}$$

when t_1 remains finite, or rather, more strictly, when $t-t_1 \to \infty$. This means that also for the non-Markovian process the events separated by very long times become uncorrelated. This result is the direct consequence of the assumption of exponential damping of the memory kernel.

These formulas are valid for ordered time sequences $t \geq t_1 \geq \cdots \geq t_n$ only. It can be shown that choosing the initial condition at some time t_0 earlier than at least one of the time moments from the set $\{t_1, \ldots, t_n\}$ leads to results incompatible with the formulas above. This property is related to the non-Markovian character of the process $\xi(t)$.

B. Averages

Consider the quantities

$$\langle \xi(t_1) \cdots \xi(t_n) \rangle = \sum_{\xi_1} \cdots \sum_{\xi_n} \xi_1 \cdots \xi_n P_n(\xi_1, t_1, \ldots, \xi_n, t_n). \tag{3.10}$$

For finite times we get from (3.5) the recurrence formula:

$$\langle \xi(t_1) \cdots \xi(t_n) \rangle = \Delta^2 \psi(t_1 - t_2) K_{n-1}(t_2, \ldots, t_n), \tag{3.11}$$

with

$$K_{n-1}(t_2, \ldots, t_n) = \langle \xi(t_3) \cdots \xi(t_n) \rangle + \Delta_0 \psi(t_2 - t_3) K_{n-2}(t_3, \ldots, t_n). \tag{3.12}$$

Note that for symmetric DN, $\Delta_0 = 0$, we get

$$\begin{aligned} \langle \xi(t_1) \cdots \xi(t_{2n}) \rangle &= \Delta^2 \psi(t_1 - t_2) \Delta^2 \psi(t_3 - t_4) \cdots \Delta^2 \psi(t_{2n-1} - t_{2n}) \\ &= \langle \xi(t_1)\xi(t_2) \rangle \langle \xi(t_3)\xi(t_4) \rangle \cdots \langle \xi(t_{2n-1})\xi(t_{2n}) \rangle , \end{aligned}$$

$$\langle \xi(t_1) \cdots \xi(t_{2n+1}) \rangle = 0 ,$$

valid only for ordered time sequence $t_1 \geq t_2 \geq \cdots \geq t_n$. For $t_1 \to \infty$, t_2 remaining finite, this implies that

$$\lim_{t_1 \to \infty} \langle \xi(t_1)\xi(t_2) \cdots \xi(t_n) \rangle = 0. \tag{3.13}$$

One of the corollaries of these results is that

$$\frac{\partial}{\partial t} \langle \xi(t)\xi(t_1) \cdots \xi(t_n) \rangle = -\chi(t - t_1) \langle \xi(t)\xi(t_1) \cdots \xi(t_n) \rangle , \tag{3.14}$$

where

$$\chi(t) = -\dot{\psi}(t)/\psi(t) = \frac{\theta_2(\theta_2 - \nu) - \theta_1(\theta_1 - \nu)e^{-\Gamma t}}{\theta_2 - \nu - (\theta_1 - \nu)e^{-\Gamma t}}. \tag{3.15}$$

Every average of the type $\langle F(\xi(t_1))G(\xi(t_2)) \cdots \rangle$ can be expressed by averages of the type (3.10) by virtue of (2.1). The only condition is that the functions $F, G, \ldots$, can be expanded into power series of their arguments. The averages of functionals $F(...;[\xi])$ of the process $\xi(t)$ can be expanded into series of averages (3.10), too

$$F(...;[\xi]) = F(...;[0]) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n \times K_n(\ldots; t_1, \ldots, t_n) \xi(t_1) \cdots \xi(t_n), \tag{3.16}$$

where

$$K_n(\ldots; t_1, \ldots, t_n) = \left(\frac{\delta^n F(...;[\xi])}{\delta\xi(t_1) \cdots \delta\xi(t_n)} \right)_{\xi=0}, \tag{3.17}$$

and where both $F(\ldots;[0])$ and $K_n(\ldots; t_1, \ldots, t_n)$ do not depend on ξ.

Therefore, the expressions (3.11)–(3.13) determine wholly all averages of interest of quantities related to the non-Markovian dichotomic process $\xi(t)$.

IV. STOCHASTIC FLOWS DRIVEN BY NON-MARKOVIAN NOISES

A. DN-driven processes

We shall consider general one-dimensional stochastic flows of the type

$$\dot{X} = f(X) + g(X)\xi(t). \tag{4.1}$$

Generalization for many-dimensional flows is trivial (formally). For dichotomic noises, flows with the right-hand side given by more general form $F(X(t), \xi(t))$ reduce to flows of the form (4.1) above due to the property (2.1). Such more general forms are meaningful, in general, for colored noises only. Nonlinear functions of white noises are ill defined: the white noise is equivalent to a series of δ functions [4], which means that the square (and higher powers) of white noise is meaningless.

Let $P(x,t)$ denote the probability density that at time interval $(t, t+dt)$ the value of the process $X(t)$ lies in the interval $(x, x+dx)$ and let $p(x, \xi_i, t)$ be the joint probability density that $X(t) \in (x, x+dx)$ and $\xi(t) = \xi_i$:

$$P(x,t) = \langle \delta(X(t,[\xi]) - x) \rangle , \tag{4.2}$$

$$p(x, \xi_i, t) = \langle \delta(X(t,[\xi]) - x)\delta(\xi(t), \xi_i) \rangle , \tag{4.3}$$

$$P(x,t) = p(x, \Delta_1, t) + p(x, -\Delta_2, t). \tag{4.4}$$

Therefore [compare with [2,9,11], and with our Eq. (2.4)]

$$\frac{\partial}{\partial t}p(x,\Delta_1,t) = -\frac{\partial}{\partial x}[f(x)+\Delta_1 g(x)]p(x,\Delta_1,t) - \gamma_0[\lambda_1 p(x,\Delta_1,t) - \lambda_2 p(x,-\Delta_2,t)]$$
$$-\gamma_1\int_{t_0}^{t} dt' e^{-\nu(t-t')}[\lambda_1 h(x,t;\Delta_1,t') - \lambda_2 h(x,t;-\Delta_2,t')]\,,$$
$$\frac{\partial}{\partial t}p(x,-\Delta_2,t) = -\frac{\partial}{\partial x}[f(x)-\Delta_2 g(x)]p(x,-\Delta_2,t) + \gamma_0[\lambda_1 p(x,\Delta_1,t) - \lambda_2 p(x,-\Delta_2,t)]$$
$$+\gamma_1\int_{t_0}^{t} dt' e^{-\nu(t-t')}[\lambda_1 h(x,t;\Delta_1,t') - \lambda_2 h(x,t;-\Delta_2,t')]\,, \quad (4.5)$$

where

$$h(x,t;\xi_i,t') = \langle\delta(X(t,[\xi]) - x)\delta(\xi(t'),\xi_i)\rangle\,, \quad (4.6)$$

and obviously

$$P(x,t) = h(x,t;\Delta_1,t') + h(x,t;-\Delta_2,t')\,. \quad (4.7)$$

This means that for the non-Markovian case the standard procedure [2,9,11] does not lead to a closed set of equations describing the probability densities of interest. Some approximations are necessary, concerning the auxiliary functions h above.

The auxiliary functions $h(x,t;\xi_i,t')$ can be removed from Eqs. (4.5) by means of the systematic expansion:

$$h(x,t;\xi_i,t') = p(x,\xi_i,t') + \frac{\partial}{\partial t}h(x,t;\xi_i,t')_{t=t'}(t-t') + \cdots\,. \quad (4.8)$$

This is in fact the short-memory (small $1/\nu$) expansion [cf., e.g., the stationary correction factors (4.35) below]. The lowest-order approximation

$$h(x,t;\xi_i,t') \approx p(x,\xi_i,t') \quad (4.9)$$

may be viewed also as suggested by the fact that both (4.4) and (4.7) hold simultaneously.

Let us begin with the approximation (4.9). In this case, the elimination of $p(x,-\Delta_2,t)$ leads to

$$\frac{\partial}{\partial t}p(x,\Delta_1,t) = -\frac{\partial}{\partial x}[f(x)+\Delta_1 g(x)]p(x,\Delta_1,t) - \int_{t_0}^{t} dt' K(t-t')[\Lambda p(x,\Delta_1,t') - \lambda_2 P(x,t')]\,, \quad (4.10)$$

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}[f(x)-\Delta_2 g(x)]P(x,t) - w_0\Lambda\frac{\partial}{\partial x}g(x)p(x,\Delta_1,t)\,, \quad (4.11)$$

and the elimination of the memory integral to

$$\hat{\mathcal{D}}(t)\frac{\partial}{\partial t}p(x,\Delta_1,t) = -\hat{\mathcal{D}}(t)\left\{\gamma_0\Lambda + \frac{\partial}{\partial x}[f(x)+\Delta_1 g(x)]\right\}p(x,\Delta_1,t) - \gamma_1\Lambda p(x,\Delta_1,t) + [\gamma_1 + \hat{\mathcal{D}}(t)\gamma_0]\lambda_2 P(x,t), \quad (4.12)$$

where $\hat{\mathcal{D}}(t) = \nu + (\partial/\partial t)$. This form is convenient for finding the stationary solution (cf. below). The formal solution to Eq. (4.10) reads

$$p(x,\Delta_1,t) = e^{-\hat{\mathcal{R}}(x,t,t_0)}p(x,\Delta_1,t_0) + \lambda_2\int_{t_0}^{t} dt' e^{-\hat{\mathcal{R}}(x,t,t')} \times[\gamma_0 + \gamma_1\hat{\mathcal{I}}(t')]P(x,t')\,, \quad (4.13)$$

where

$$\hat{\mathcal{R}}(x,t,t_0) = \left\{\gamma_0\Lambda + \frac{\partial}{\partial x}[f(x)+\Delta_1 g(x)]\right\}(t-t_0) + \gamma_1\Lambda\int_{t_0}^{t} dt'\hat{\mathcal{I}}(t')\,, \quad (4.14)$$

and

$$\hat{\mathcal{I}}(t,t_0) = \hat{\mathcal{I}}(t) = \hat{\mathcal{D}}(t)^{-1} = \int_{t_0}^{t} dt' e^{-\nu(t-t')}\,, \quad (4.15)$$

i.e.,

$$\int_{t_0}^{t} dt' K(t-t') = \gamma_0 + \gamma_1\hat{\mathcal{I}}(t)\,. \quad (4.16)$$

This is the formal solution. Substitution to Eq. (4.11) gives the formally closed equation for the probability density $P(x,t)$. Putting $\gamma_1 = 0$ and $p(x,\Delta_1,t_0) = 0$ (after [2,9,11]) we recover the formulas of Refs. [2,9,11].

Taking into account the first correction in (4.8) we get Eq. (4.10) with the following additional term on the right-hand side [Eq. (4.11) remains unchanged]:

$$+\gamma_1\frac{\partial}{\partial x}\int_{t_0}^{t} dt'(t-t')e^{-\nu(t-t')}\{[\Lambda f(x) + (\lambda_1\Delta_1 - \lambda_2\Delta_2)g(x)]p(x,\Delta_1,t') - \lambda_2[f(x)-\Delta_2 g(x)]P(x,t')\}\,. \quad (4.17)$$

Therefore, Eq. (4.12) now reads

$$[\hat{\mathcal{D}}(t)]^2 \frac{\partial}{\partial t} p(x,\Delta_1,t) = -[\hat{\mathcal{D}}(t)]^2 \left\{ \gamma_0 \Lambda + \frac{\partial}{\partial x}[f(x) + \Delta_1 g(x)] \right\} p(x,\Delta_1,t) + [\hat{\mathcal{D}}(t)]^2 \gamma_0 \lambda_2 P(x,t) - \gamma_1 \hat{\mathcal{D}}(t)[\Lambda p(x,\Delta_1,t) - \lambda_2 P(x,t)] + \gamma_1 \frac{\partial}{\partial x}\{[\Lambda f(x) + (\lambda_1\Delta_1 - \lambda_2\Delta_2) g(x)] p(x,\Delta_1,t) - \lambda_2 [f(x) - \Delta_2 g(x)] P(x,t)\} . \quad (4.18)$$

Other possible simple approximations are based on the shifting of the time dependence of the auxiliary function h by the function $\psi(t-t')$, which is suggested by the results of Secs. II and III. Therefore, we may write that either

$$h(x,t;\xi,t') \approx \Delta^2 \psi(t-t') p(x,\xi,t') \quad (4.19)$$

(factor Δ^2 is introduced in order to keep correct WN limit) or, inversely,

$$h(x,t;\xi,t') \approx [\Delta^2 \psi(t-t')]^{-1} p(x,\xi,t) . \quad (4.20)$$

However, these approximations are not systematic, but rather constitute a kind of *ansatzes*. In the case of approximation (4.19) we get again Eq. (4.10), with the changed kernel:

$$K(t-t') \to \tilde{K}(t-t') = \gamma_0 \delta(t-t') + \gamma_1 e^{-\nu(t-t')} \Delta^2 \psi(t-t') , \quad (4.21)$$

and thus, after elimination of memory integral,

$$\hat{\mathcal{D}}_1(t)\hat{\mathcal{D}}_2(t) \frac{\partial}{\partial t} p(x,\Delta_1,t) = -\hat{\mathcal{D}}_1(t)\hat{\mathcal{D}}_2(t) \frac{\partial}{\partial x}[f(x) + \Delta_1 g(x)] p(x,\Delta_1,t) \hat{\mathcal{D}}_1(t)\hat{\mathcal{D}}_2(t) \gamma_0 [\Lambda p(x,\Delta_1,t) \lambda_2 P(x,t)] - \frac{\gamma_1}{\Gamma}[(\theta_1 - \nu)\hat{\mathcal{D}}_2(t) - (\theta_2 - \nu)\hat{\mathcal{D}}_1(t)][\Lambda p(x,\Delta_1,t) - \lambda_2 P(x,t)] , \quad (4.22)$$

where $\hat{\mathcal{D}}_i(t) = \theta_i + \hat{\mathcal{D}}(t)$.

The approximation (4.20) leads to the equation

$$\frac{\partial}{\partial t} p(x,\Delta_1,t) = -\frac{\partial}{\partial x}[f(x) + \Delta_1 g(x)] p(x,\Delta_1,t) - [\gamma_0 + \gamma_1 \Omega(t)][\Lambda p(x,\Delta_1,t) - \lambda_2 P(x,t)] , \quad (4.23)$$

with

$$\Omega(t) = \int_{t_0}^{t} dt' e^{-\nu(t-t')} [\Delta^2 \psi(t-t')]^{-1} . \quad (4.24)$$

In this case the master equation does not exhibit explicit "memory;" instead, it contains time-dependent coefficient at its right-hand side.

The calculations illustrating the advantages and disadvantages of these approximations are presented in the Appendix C.

Approximation (4.19), though nonsystematic, seems to be natural enough, especially in the light of the formula (3.11). Indeed, the results presented in Appendix C suggest that (4.19) constitutes the distinct improvement over the approximation (4.9), being at the same time much simpler than the higher-order systematic approximations stemming from the expansion (4.8). The sole reason for the introduction of the approximation (4.20) is that it seems to be just the inverse of (4.19). Nevertheless, the results obtained in Appendix C suggest that this approximation leads to too fast diffusion of probability density and thus may be reliable at best at initial stages of the evolution of $P(x,t)$.

B. Telegrapher's equation

For $f(x) = 0$, $g(x) = 1$ the stochastic process $X(t)$ becomes just the integral of the process $\xi(t)$. When $\xi(t)$ is the symmetric Markovian dichotomic noise, the equation for the probability density $P(x,t)$ fulfills the well-known telegrapher's equation. When $\xi(t)$ becomes the non-Markovian DN, the appropriate equation, obtained from Eqs. (4.10), (4.11) reads in the lowest-order approximation (4.9):

$$\left(\frac{\partial^2}{\partial t^2} + \Delta_0 \frac{\partial^2}{\partial t \partial x} - \Delta^2 \frac{\partial^2}{\partial x^2} + \gamma_0 \Lambda \frac{\partial}{\partial t} \right) P(x,t) = -\gamma_1 \Lambda \int_{t_0}^{t} dt' e^{-\nu(t-t')} \frac{\partial}{\partial t'} P(x,t') . \quad (4.25)$$

The left-hand side gives the standard telegrapher's equation with the asymmetric term proportional to Δ_0, whereas the right-hand side is the non-Markovian correction. Note that the term containing first-order time derivative is here of purely Markovian origin.

Approximation (4.18) leads to higher-order equations, approximation (4.21) changes the kernel in Eq. (4.25), whereas the approximation (4.23) gives the telegrapher's equation with time-dependent coefficient of the last term:

$$\left(\frac{\partial^2}{\partial t^2}+\Delta_0\frac{\partial^2}{\partial t\partial x}-\Delta^2\frac{\partial^2}{\partial x^2}+[\gamma_0+\gamma_1\Omega(t)]\Lambda\frac{\partial}{\partial t}\right)P(x,t)=0\,. \quad (4.26)$$

C. WN-driven processes

We shall write here explicitly only formulas corresponding to the first approximation (4.12). Formulas resulting from other approximations can be derived in the same manner. In the WSN limit (2.14) we get [2] (cf. Appendix A):

$$\frac{\partial}{\partial t}P(x,t)=-\frac{\partial}{\partial x}\left[f(x)-\Delta_2 g(x)\right]P(x,t)-w_0\lambda_2\frac{\partial}{\partial x}g(x)\left[\gamma_0+\gamma_1\hat{\mathcal{I}}(t)+w_0\frac{\partial}{\partial x}g(x)\right]^{-1}\times\int_{t_0}^{t}dt'K(t-t')P(x,t')\,, \quad (4.27)$$

which corresponds to Eq. (32) of Ref. [2].

The GWN limit is easy to obtain: the limit (2.15) is equivalent to

$$w_0\to 0\,,\quad \lambda_2\to\infty\,,$$
$$w_0^2\lambda_2=D_0\,,\quad w_0\lambda_2=\Delta_2\to\infty\,. \quad (4.28)$$

Therefore, from (4.27),

$$\frac{\partial}{\partial t}P(x,t)=-\frac{\partial}{\partial x}f(x)P(x,t)+D_0\frac{\partial}{\partial x}g(x)\frac{\partial}{\partial x}g(x)\times\left[\int_{t_0}^{t}dt'K(t-t')\right]^{-1}P(x,t)\,. \quad (4.29)$$

This is the non-Markovian Fokker-Planck equation in lowest-order approximation (4.12). Again, approximations (4.18) and (4.21) lead to higher-order equations. In the approximation (4.23) the last integral in Eqs. (4.27) and (4.29) is replaced by the factor $[\gamma_0+\gamma_1\Omega(t)]P(x,t)$.

The left-hand side of the telegrapher's equation (4.25) reads in these limits

$$\left[\frac{\partial}{\partial t}\left(\gamma_0+\alpha\frac{\partial}{\partial x}\right)-\sigma\frac{\partial^2}{\partial x^2}\right]P(x,t)\,, \quad (4.30)$$

with $\alpha=0$ for GWN, and $\alpha=w_0$ for WSN, and the right-hand side remains the same as in Eq. (4.25) divided by Λ.

D. Stationary distributions

Stationary solutions to the above equations can be obtained in a simplest way by putting $P(x,t)\to P_{st}(x)$, $p(x,\Delta_1,t)\to p_{st}(x,\Delta_1)$ in Eqs. (4.11), (4.12), (4.18), (4.21), or (4.23). From (4.11) we get

$$\frac{\partial}{\partial x}[f(x)-\Delta_2 g(x)]P_{st}(x)=-w_0\Lambda\frac{\partial}{\partial x}g(x)p_{st}(x,\Delta_1)\,. \quad (4.31)$$

The approximation (4.12) leads to

$$\zeta\lambda_2 P_{st}(x)=\left\{\zeta\Lambda+\frac{\partial}{\partial x}[f(x)+\Delta_1 g(x)]\right\}p_{st}(x,\Delta_1)\,. \quad (4.32)$$

Similar expressions are obtained for the remaining approximations. Therefore, with natural boundary conditions, the stationary distribution reads

$$P_{st}(x)=\mathcal{N}^{-1}\frac{|g(x)|}{D_{\rm eff}(x)}\times\exp\left[\Lambda\zeta\int^{x}dx\frac{f(x)}{D_{\rm eff}(x)}\right]\Theta(D_{\rm eff}(x))\,, \quad (4.33)$$

where

$$D_{\rm eff}(x)=[\Delta_1 g(x)+f(x)][\Delta_2 g(x)-f(x)]\,, \quad (4.34)$$

$\mathcal{N}$ is the normalization constant, and $\Theta(x)$ is the Heaviside step function, "expressing that the probability is zero in the "unstable" region of negative D" [11]. This expression differs from the non-Markovian case by the rescaling $\Lambda\to\zeta\Lambda$ only. Therefore, non-Markovian approximations (4.12), (4.21), and (4.23) leave the domain $\mathcal{D}_x$ in which $P_{st}(x)$ is nonzero unchanged in comparison with the Markovian case. Both white noise limits are thus given by Markovian stationary distributions with the same rescaling, too.

The rescaling parameters are (indexes refer to approximations)

$$\zeta_{12}=\gamma_0+\gamma_1/\nu\,,\quad \zeta_{18}=\nu(\nu\gamma_0+\Lambda\gamma_1)/(\nu^2-\Lambda\gamma_1)\,,$$
$$\zeta_{22}=\gamma_0+2\gamma_1\nu/(\nu+\theta_1)(\nu+\theta_2)\,,\quad \zeta_{23}=\gamma_0+\gamma_1\Omega(\infty)\,. \quad (4.35)$$

The renormalization of the magnitude of the correlation time Λ^{-1} seems to be insignificant: correlation times of the noises are not known usually. The change of the sign of $\zeta\Lambda$ may, however, change the location of extrema of P_{st}, i.e., the most probable values of x_{st}. The analysis of the stability conditions for the dichotomic process $\xi(t)$ leads to the conclusion that the conditions $\mathrm{Re}(\theta_1)>0$, $\mathrm{Re}(\theta_2)>0$ demand that $\zeta_{12}>0$. Therefore, the approximation (4.12) does not change stationary properties. The rescaling parameters ζ_{18}, ζ_{22}, and ζ_{23} may become negative, so that these approximations allow for changes of stationary characteristics of the process $\dot{X}$ resulting from the non-Markovian character of the driving noise.

V. AN EXAMPLE: ORNSTEIN-UHLENBECK PROCESS

A. Direct calculations

One of the simplest examples of stochastic flows is the linear relaxation process driven by additive noise:

$$\dot{X}(t) = -aX(t) + b\xi(t)\,, \quad a > 0\,. \tag{5.1}$$

This process may represent, among others, the velocity of the damped Brownian particle, or the position of the overdamped Brownian particle. When $\xi(t)$ is the Markovian GWN, the process (5.1) is called the Ornstein-Uhlenbeck (OU) process and is the well-known Markovian colored noise (CGN), widely used in literature, with stationary correlation function:

$$\langle X(t)X(t')\rangle = (b^2D_0/2a)e^{-a|t-t'|}\,. \tag{5.2}$$

Consider now the same process driven by non-Markovian noise. The calculation of averages of the type $\langle X^m\rangle$ from the stationary distribution (5.12) below is possible but very tedious (the appropriate integrals lead to hypergeometric functions). Moreover, the calculation of time-dependent averages from Eqs. (4.10) and (4.11) cannot be practically performed. Such average values are simpler to calculate directly. Besides, the equations of the preceding section are but approximations.

The solution of (5.1), and its correlation function read

$$X(t) = e^{-a(t-t_0)}X(t_0) + b\int_{t_0}^{t} dt'\,\xi(t')\,e^{-a(t-t')}\,, \tag{5.3}$$

$$\begin{aligned}\langle X(t)X(t')\rangle - \langle X(t)\rangle\langle X(t')\rangle &= \int_{t_0}^{t} dt_1 e^{-a(t-t_1)}\int_{t_0}^{t'} dt_2 e^{-a(t'-t_2)}\,C_{st}(t_1,t_2)\\ &= (b^2\Delta^2/\Gamma)[(\theta_1-\nu)J(t,t';\theta_1) - (\theta_2-\nu)J(t,t';\theta_2)] \quad \text{for DN} &(5.4)\\ &= \frac{b^2\sigma}{2a\gamma_0}[e^{-a|t-t'|} - e^{-a(t+t')}] - \frac{b^2\sigma\gamma_1}{\gamma_0^2}J(t,t';\theta_0) \quad \text{for WN}\,, &(5.5)\end{aligned}$$

where

$$J(t,t';\theta) = \frac{1}{a(a^2-\theta^2)}\left[ae^{-\theta|t-t'|} - \theta e^{-a|t-t'|} + (a+\theta)e^{-a(t+t')} - a\left(e^{-at'-\theta t} + e^{-at-\theta t'}\right)\right]. \tag{5.6}$$

The variance $\langle X^2(t)\rangle$ and corresponding stationary formulas are easy to obtain from the above. In the DN case the correlation function and variance may oscillate.

The process (5.1) driven by non-Markovian GWN corresponds to the non-Markovian Ornstein-Uhlenbeck process, characterized by the following stationary correlation function:

$$\langle XX(\tau)\rangle_{st} = \frac{b^2D_0}{a\gamma_0}\left[\frac{1}{2}e^{-a\tau} - \frac{\gamma_1}{a^2-\theta_0^2}\times\left(ae^{-\theta_0\tau} - \theta_0 e^{-a\tau}\right)\right]. \tag{5.7}$$

This relation defines the stationary non-Markovian colored Gaussian noise (GCN).

An interesting quantity seems to be the stationary variance, which reads

$$\langle X^2\rangle_{st} = \frac{b^2\Delta^2(a+\nu)}{a(a+\theta_1)(a+\theta_2)}\,, \tag{5.8}$$

for the DN-driven process and for the WN-driven process:

$$\langle X^2\rangle_{st} = \frac{b^2\sigma}{a\gamma_0}\left(\frac{1}{2} - \frac{\gamma_1}{a+\theta_0}\right). \tag{5.9}$$

These formulas enable to find whether the driving process is Markovian or non-Markovian by looking for the dependence of the stationary variance on the parameter a of the deterministic process $X(t)$. It is easy to find that

$$a\langle X^2\rangle_{st} \sim \begin{cases} 1 & \text{for Markovian WN,}\\ 1-\frac{2\gamma_1}{a+\theta_0} & \text{for non-Markovian WN,}\end{cases} \tag{5.10}$$

and

$$a(a+C)\langle X^2\rangle_{st} \sim \begin{cases} 1 & \text{for Markovian DN,}\\ 1+\frac{\nu-\theta_2}{a+\theta_2} & \text{for non-Markovian DN.}\end{cases} \tag{5.11}$$

Hence for the Markovian white noise the quantity $a\langle X^2\rangle_{st}$ does not depend on a, whereas for non-Markovian white noise it does; for the Markovian dichotomic noise the quantity $a(a+C)\langle X^2\rangle_{st}$ does not depend on a, whereas for non-Markovian DN it does. Therefore, it is possible to distinguish between these types of noise, and also to determine the values of relevant noise parameters, by measuring the stationary variance as the function of the deterministic parameter a. Similar though more involved analysis can be done for time-dependent variance and for stationary and nonstationary correlation function.

The quantity $X(t)$ in Eq. (5.1) can be interpreted also as the velocity in the second-order process, which in turn can be interpreted as the (one-dimensional) damped Brownian motion. For undamped Brownian motion, i.e., for $a = 0$ the time-dependent expressions contain terms growing unboundedly with time. Thus, for example, the asymptotic in time velocity dispersion grows unboundedly as t (more details can be found in the Appendix B) and position dispersion as t^3. This behavior, identical as in the Markovian case, is called anomalous diffusion. The equation for the velocity probability density is just the non-Markovian telegrapher's equation, Eq. (4.25) above. The equation for the position density can be obtained by methods constructed recently by Masoliver [31], but the resulting expressions are rather lengthy and not very illuminating.

B. Comparison with results from $P_{st}(x)$

The approximations considered in the Sec. IV lead to the following stationary expressions:

$$D_{\text{eff}}(x) = (\Delta_1 - ax)(\Delta_2 + ax)\,, \tag{5.12}$$

$$P_{st}(x) = \mathcal{N}^{-1}|\Delta_1 - ax|^{\alpha_1 - 1}|\Delta_2 + ax|^{\alpha_2 - 1}\Theta(D_{\text{eff}}(x))\,,$$

$$\alpha_i = \Delta_i\Lambda\zeta/a(\Delta_1 + \Delta_2)\,. \tag{5.13}$$

Approximations (4.12), (4.21), and (4.23) suggest that $P_{st}(x)$ is nonzero for $x \in (-\Delta_2/a\,,\,\Delta_1/a)$, and $P_{st}(x) = 0$ otherwise. For approximation (4.12), for which $\zeta > 0$, the general properties of the stationary distribution remain the same as in the Markovian case, with one exception: for the marginal point $\gamma_1 = -\nu\gamma_0$ we have $\zeta = 0$, $\theta_2 = 0$, $\alpha_i = 0$, and the whole density contracts towards both accumulation points $x_- = -\Delta_2/a$, $x_+ = \Delta_1/a$. Other approximations may change the general shape of $P_{st}(x)$, particularly the location of most probable values of x_{st}. Note that the extremal points $-\Delta_2/a$, $+\Delta_1/a$ of the distribution are attracting only when $\alpha_1 < 1$, $\alpha_2 < 1$, respectively. Therefore, $\zeta < 0$ implies that these points are attracting always, regardless of the values of remaining noise parameters.

These approximations lead to the stationary variance of the same form as for linear process driven by Markovian DN:

$$\langle X^2\rangle_{st} = b^2\Delta^2/a(a + \zeta\Lambda)\,, \tag{5.14}$$

in disagreement with exact result (5.8) above. Moreover, for $\zeta < 0$ these approximations predict strong increase of variance, which also seems to be incorrect.

VI. FINAL REMARKS AND CONCLUSIONS

We have defined the non-Markovian dichotomic noise by means of the master equations (2.4) containing explicit, exponentially damped memory. The correlation function of such noise is characterized by two different correlation times and for some combinations of the noise parameters may become damped oscillatory.

White non-Markovian noises are defined as limits of the dichotomic noise. The limiting procedures are well defined only when there is the Markovian admixture in the non-Markovian kernel of the master equation. In this way the non-Markovian Fokker-Planck equation is derived. Its form is different from that proposed by Risken [32]. The latter is written *ad hoc* with the whole right-hand side under non-Markovian memory integral.

As we have mentioned above, non-Markovian white noises discussed in this paper are characterized by nonzero correlation time θ_0^{-1}. Therefore, one may argue that the use of the term "white noise" for such entities is self-contradictory. It can be also argued that the term "non-Markovian white noise" itself is contradictory. In present author's opinion this is largely the question of definition what is "white noise." If we agree that this is the uncorrelated (δ-function-correlated) process, or the process "without memory," then indeed the process $\xi(t)$ in the limits (2.14) and (2.15) will not be white. However, if we agree that white process is that composed of δ spikes, then the mentioned limits of the process $\xi(t)$ do lead to non-Markovian white noises. In this paper we have used the term "WN's" mainly for convenience, to make a distinction between dichotomic process composed of random signals of finite duration from that composed of δ-function spikes. These differences are important: in the former case $[\xi(t)]^2$ is well defined [cf. Eq. (2.1)], whereas in the latter ξ^2 has no meaning.

We have sketched four different approximations for the master equations describing the probability densities of the general process $\dot{X}(t)$ driven by non-Markovian noise. It is difficult to estimate fully their advantages and faults on the basis of the results discussed above. Some insight may be obtained from the results discussed in the Appendix C.

What seems to be particularly interesting is the differences between properties of Markovian and non-Markovian noises, especially the differences in behavior of stochastic processes driven by these noises. In that respect, the non-Markovianity of the noise $\xi(t)$ is best visible in time-dependent quantities, especially in the case of mixed non-Markovianity, when the process oscillates, i.e., when the inverse correlation times θ_i contain nonzero imaginary part. These oscillations are transferred to driven processes $X(t)$. This can be seen explicitly in time-dependent quantities considered in Sec. V.

The oscillations are possible only for a mixed process. For purely Markovian and purely non-Markovian processes the time dependence is monotonic. That is, the oscillations are the result of the competition between Markovian and non-Markovian sub-processes. Moreover, for a linear process the time-dependent quantities are characterized by different number of relaxation times: one for Markovian white noises, two for non-Markovian white noises and for Markovian dichotomic noise, and three (or two and one frequency) for non-Markovian DN's. Note that from this point of view there are no detectable differences between processes driven by GWN's and WSN's, and between Markovian DN's and non-Markovian WN's. However, the linear processes driven by non-Markovian WN's and by Markovian DN's differ in the dependence of their stationary variances on the deterministic parameter a [cf. Eqs. (5.10) and (5.11)].

The influence of non-Markovianity on stationary properties is more subtle. The stationary distributions of the process $\xi(t)$ are the same as for the Markovian process. The stationary properties of driven processes are different from the non-Markovian case, but direct calculation of these differences is possible only in a small number of simple cases (cf. Sec. V). All approximations discussed explicitly above are unable to predict correctly the properties of stationary variance of the driven linear process. Nevertheless, these approximations suggest that the non-Markovianity of the driving noise manifests itself at least in the rescaling of the correlation time Λ^{-1}. Since usually we do not know correlation times of the noise *a priori*, the rescaling of Λ is insignificant at the level of description by properties of $P_{st}(x)$: number and location of extrema, width of the distribution, etc. as long as rescaled Λ does not change its sign. However, the change of the sign of $\zeta\Lambda$ may shift the location of extrema and thus the most probable stationary values of X. Therefore, even these simple approximations may change the detailed shape of stationary distribution, i.e., may change at least some of the stationary properties of the driven system. Still, these approximations are unable to reproduce correctly for example the properties of the stationary variance of the driven linear process.

Higher-order terms of the short-memory expansion (4.8), not discussed explicitly here, lead to higher-order equations for $P_{st}(x)$. The solutions of such equations are no longer given by simple expressions of the type of Eq. (4.33). This means that higher-order approximations may change also the general form of $P_{st}(x)$ and the domain of x in which P_{st} is nonzero, i.e., change all stationary properties of the driven process. This means, in turn, that the non-Markovianity of the driving noise may change radically not only the time-dependent behavior but also the stationary properties of the driven stochastic process.

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APPENDIX A: WHITE NOISE LIMITS

We shall present here some details of the calculations leading to the formula (2.17). Applying the limits (2.14), (2.15) we get for $\gamma_0 \neq 0$

$$\Gamma = (\gamma_0\Lambda - \nu)\sqrt{1 - \tfrac{4\gamma_1\Lambda}{(\gamma_0\Lambda\nu)^2}}$$
$$= \gamma_0\Lambda_0 + s - \nu - 2\mu - \mu(\nu + 2\mu)/\gamma_0\Lambda_0 + O(\Lambda^{-2}),$$

$$\theta_1 = \gamma_0\Lambda_0 + s - \mu - \mu(\nu + 2\mu)/2\gamma_0\Lambda_0 + O(\Lambda^{-2}),$$

$$\theta_2 = \theta_0 + \mu(\nu + 2\mu)/2\gamma_0\Lambda_0 + O(\Lambda^{-2}),$$

$$\frac{\theta_1 - \nu}{\Gamma} = 1 + \frac{\mu}{\gamma_0\Lambda_0} + O(\Lambda^{-2}),$$

$$\frac{\theta_2 - \nu}{\Gamma} = \frac{\mu}{\gamma_0\Lambda_0} + O(\Lambda^{-2}), \tag{A1}$$

where $\mu = \gamma_1/\gamma_0$, $s = \gamma_0\lambda_2$ for WSN and $s = 0$ for GWN. This leads to

$$\psi(t) = \left(1 + \tfrac{\mu}{\gamma_0\Lambda_0}\right)e^{-\gamma_0\Lambda_0 t}e^{(\mu-s)t} - \tfrac{\mu}{\gamma_0\Lambda_0}e^{-\theta_0 t}. \tag{A2}$$

(A2) with (2.16) leads directly to (2.17).

For $\gamma_0 = 0$, we get in the same way

$$\psi(t) = e^{-\nu t/2}\left[\cos(\tilde{\omega}t) + \frac{\nu}{2\omega}\sin(\tilde{\omega}t)\right], \tag{A3}$$

where

$$\tilde{\omega} = \omega(1 - a/\omega^2)\,, \omega^2 = \gamma_1\Lambda \to \infty\,,$$
$$8a = \begin{cases} \nu^2 - 4\gamma_1\lambda_2 & \text{for WSN} \\ \nu^2 & \text{for GWN}. \end{cases}$$

The derivation of Eq. (4.27) is as follows:

$$\lim_{\lambda_1\to\infty}\lambda_1\exp[-\hat{\mathcal{R}}(x,t,t')] = \lim_{\lambda_1\to\infty}\lambda_1\exp\left\{-\left((\lambda_1+\lambda_2)\left[\gamma_0 + \gamma_1\hat{\mathcal{T}}(t,t')\right] + \frac{\partial}{\partial x}\left[f(x) + w_0\lambda_1 g(x)\right]\right)(t-t')\right\}$$
$$= \delta\left(\left[\gamma_0 + \gamma_1\hat{\mathcal{T}}(t,t') + w_0\frac{\partial}{\partial x}g(x)\right](t-t')\right), \tag{A4}$$

where

$$\hat{\mathcal{T}}(t,t') = \frac{1}{t-t'}\int_{t'}^{t} dt_1\hat{\mathcal{I}}(t_1,t_0). \tag{A5}$$

Then

$$
\begin{aligned}
\hat{\mathcal{T}}(t,t)h(t) &= \lim_{\tau\to 0}\frac{1}{\tau}\int_t^{t+\tau} dt_1 \hat{\mathcal{I}}(t_1)h(t_1) \\
&= \lim_{\tau\to 0}\frac{1}{\tau}\int_t^{t+\tau} dt_1 \left(\nu+\frac{\partial}{\partial t}\right)^{-1} h(t_1) \\
&= \lim_{\tau\to 0}\frac{1}{\tau\nu}\int_t^{t+\tau} dt_1 \left[1-\frac{\partial}{\nu\partial t_1}+\left(\frac{\partial}{\nu\partial t_1}\right)^2-\dots\right]h(t_1) \\
&= \lim_{\tau\to 0}\frac{1}{\tau\nu}\left\{[H(t+\tau)-H(t)]-\frac{1}{\nu}[h(t+\tau)-h(t)]+\frac{1}{\nu^2}[\dot{h}(t+\tau)-\dot{h}(t)]-\dots\right\} \\
&= \frac{1}{\nu}\left[h(t)-\frac{1}{\nu}\dot{h}(t)+\frac{1}{\nu^2}\ddot{h}(t)-\dots\right] = \left(\nu+\frac{\partial}{\partial t}\right)^{-1} h(t),
\end{aligned}
$$

which implies

$$\hat{\mathcal{T}}(t,t) = \hat{\mathcal{I}}(t)\,. \tag{A6}$$

Therefore,

$$\lim_{\lambda_1\to\infty} \lambda_1 \exp[-\hat{\mathcal{R}}(x,t,t')] = \left[\gamma_0 + w_0\frac{\partial}{\partial x}g(x) + \gamma_1\hat{\mathcal{I}}(t,t_0)\right]^{-1}\delta(t-t_0)\,, \tag{A7}$$

$$\lim_{\lambda_1\to\infty} \lambda_1 p(x,\Delta_1,t) = \left[\gamma_0 + w_0\frac{\partial}{\partial x}g(x) + \gamma_1\hat{\mathcal{I}}(t,t_0)\right]^{-1}[\gamma_0+\gamma_1\hat{\mathcal{I}}(t,t_0)]\lambda_2 P(x,t)\,. \tag{A8}$$

Recalling (4.16) and substituting these formulas to Eqs. (4.11), (4.13), we obtain Eq. (4.27).

APPENDIX B: MOMENTS OF RANDOM TELEGRAPH PROCESS

To obtain an insight into the consequences of the basic assumption of the Sec. III, let us find the properties of the moments describing the random telegraph process:

$$\dot{x}(t) = \xi(t), \tag{B1}$$

assuming for simplicity the symmetric DN: $\Delta_1 = \Delta_2 = \Delta$, $\lambda_1 = \lambda_2 = \lambda$, $\Delta_0 = 0$. It is easy to find that

$$\langle x^{2n}(t)\rangle = \sum_{m=0}^{n-1} f_{n,m}(e^{-\theta_1 t}, e^{-\theta_2 t})t^m + a_n t^n, \tag{B2}$$

with

$$
\begin{aligned}
a_n &= \frac{(2n)!}{n!}\left[\frac{\Delta^2}{\Gamma}\left(\frac{\theta_1-\nu}{\theta_1}-\frac{\theta_2-\nu}{\theta_2}\right)\right]^n \\
&= \frac{(2n)!}{n!}\left[\frac{\nu\Delta^2}{(\nu\gamma_0+\gamma_1)\Lambda}\right]^n,
\end{aligned} \tag{B3}
$$

and $f_{n,m}$ being rather complicated functions of noise parameters. Therefore, the non-Markovian DN defined in Sec. III leads to the random telegraph process roughly similar to that driven by Markovian kinetics, especially for long times, when all transients die out. There are quantitative differences, moreover, transients may become oscillatory. The asymptotic behavior, however, is governed by the same power law as in the Markovian case, though with different rate coefficients a_n: the ratio of non-Markovian to Markovian coefficients is

$$\frac{(a_n)_{\text{n-M}}}{(a_n)_{\text{Mark}}} = (1+\mu/\nu)^{-n}. \tag{B4}$$

Therefore, the non-Markovianity slows down the spread off rate when $\mu = \gamma_1/\gamma_0 > 0$, and enhances (unboundedly) the spread off when Markovian and non-Markovian contributions are of opposite sign.

The moments of the second-order process, mentioned at the end of Sec. V, grow as t^{3n}.

APPENDIX C: CHECK OF APPROXIMATIONS OF SECTION IV

In this appendix we shall compare various approximations for the probability density $P(x,t)$ for the random telegraph process (C.1). In this case, $P(x,t)$ is governed by various versions of the telegrapher's equation: Eq. (4.25) in the approximation (4.9), Eq. (4.25) with the kernel containing the factor $\Delta^2\psi(t-t')$ in the approximation (4.19), or Eq. (4.26) in the approximation (4.20). Equation (4.25) with the right-hand side put equal to zero describes the Markovian case.

The solution to all these equations, subject to the initial condition:

$$P(x,0) = \delta(x) \tag{C1}$$

can be written (for $\Delta_0 = 0$) as

$$P(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, \cos(kx)\, T(t,k) \tag{C2}$$

with $T(t=0,k) = 1$, $T(t,k=0) = 1$ representing the initial condition and normalization, respectively.

The even moments (all odd moments vanish) of $P(t,k)$ can be calculated from the formula

$$\langle x^{2n}(t)\rangle = (-1)^n \frac{\partial^{2n}}{\partial k^{2n}} T(t,k)\Big|_{k=0}. \tag{C3}$$

To simplify the calculations and the discussion as far as possible, we shall consider only the white noise limit. In this case,

$$\Delta^2\psi(t) = D_0[\delta(t) - \mu e^{-\theta_0 t}], \tag{C4}$$

(cf. Appendix A) with $\theta_0 = \mu + \nu$, $\mu = \gamma_1/\gamma_0$, which leads to

$$\Omega(t) = -(e^{\mu t} - 1)/D_0\mu^2, \tag{C5}$$

(here, $D_0 = \Delta^2/2\lambda\gamma_0$) and the telegrapher's equation becomes the Fokker-Planck equation. Therefore, the function $T(t,k)$ fulfills the equation

$$(\tfrac{d}{dt} + D_0k^2)T(t,k) = \mathcal{A}(T) \tag{C6}$$

with $\mathcal{A} = 0$ for the Markovian case,

$$\mathcal{A} = \mu\int_0^t dt'\, \tilde{K}(t-t')\frac{d}{dt'}T(t'), \tag{C7}$$

with $\tilde{K}(t) = \exp(-\nu t)$ for the approximation (4.9), $\tilde{K}(t) = \exp(-\nu t)\Delta^2\psi(t)$ for the approximation (4.19), and

$$\mathcal{A} = -\mu\Omega(t)\tfrac{d}{dt}T(t) \tag{C8}$$

for the approximation (4.20). The solutions of these equations read

$$T(t,k) = \beta_1 e^{-\alpha_1} + \beta_2 e^{-\alpha_2} \tag{C9}$$

for the approximations (4.9) and (4.19), and

$$T(t,k) = e^{-D_0k^2\varphi(t)} \tag{C10}$$

for the approximation (4.20) and in the Markovian case. In the latter case $\varphi(t) = t$, whereas for (4.20)

$$\varphi(t) = \alpha_3\left[t - \tfrac{1}{\mu}\ln\left|\beta_3(e^{\mu t} - 1)\right|\right], \tag{C11}$$

and

$$\alpha_3 = D_0\mu^2/\gamma_0(1 + D_0\mu), \quad \beta_3 = \gamma_0^3 D_0/\mu, \tag{C12}$$

$$\beta_1 = 1 - \beta_2 = \frac{\alpha_2(\alpha_1 - \bar{\nu})}{\bar{\nu}(\alpha_1 - \alpha_2)}, \tag{C13}$$

$$\alpha_{1,2} = \tfrac{1}{2}\left(\alpha_0 \pm \sqrt{\alpha_0^2 - 4D\bar{\nu}k^2}\right), \quad \alpha_0 = Dk^2 + \bar{\nu} - g, \tag{C14}$$

with $D = D_0$, $\bar{\nu} = \nu$, $g = \mu$ for the approximation (4.9), and

$$D = D_0/(1 - \mu D_0), \quad \bar{\nu} = \nu + \theta_0, \quad g = -\mu^2 D \tag{C15}$$

for the approximation (4.19).

Therefore, approximations (4.9) and (4.19) are in this case rather similar—they differ mainly in the detailed dependence on various noise parameters. The approximation (4.19) seems to be more "stiff" than the approximation (4.9): for (4.9) $\alpha_{1,2}$ become complex conjugate when $\mu > \nu$, whereas for (4.19)—when

$$-\tfrac{1}{2D}(1 + \sqrt{1 - 8\nu D}) < \mu < -\tfrac{1}{2D}(1 - \sqrt{1 - 8\nu D}), \tag{C16}$$

and $8\nu D < 1$, which gives much more narrow range of μ allowing for oscillatory behavior than (4.9).

The probability density can be expressed in terms of elementary functions only for the Markovian case, and for the approximation (4.20)

$$P(x,t) = [4\pi D_0\varphi(t)]^{-1/2} e^{-x^2/4D_0\varphi(t)}. \tag{C17}$$

Nevertheless, the moments can be calculated direcly for all considered approximations. We present results for the second moment, i.e., for the time-dependent dispersion $\langle x^2(t)\rangle^{1/2} = D_x(t)$:

$$D_x^2(t) = 2\alpha_3\varphi(t) \tag{C18}$$

for the approximation (4.20) (for the Markovian case $D_x^2(t) = 2D_0t$), and

$$D_x^2(t) = \frac{2D}{\bar{\nu} - g}\left\{\bar{\nu}t - \frac{g}{\bar{\nu} - g}\left[1 - e^{-(\bar{\nu}-g)t}\right]\right\}. \tag{C19}$$

These results are to be compared with the exact results, obtained by direct calculation in the Appendix B. In that light the most suspect seems to be the approximation (4.20), which predicts the spreading of the probability density over the whole real axis, and the infinite value of the dispersion after a finite time

$$t^* = \tfrac{1}{\mu}\ln\left|1 + \tfrac{1}{\mu}\right|. \tag{C20}$$

Therefore, the approximation (4.20) might be useful for short times only.

In the approximation (4.19) $\bar{\nu} - g = 2\nu + \mu + \mu^2 D$, and thus the growth of the dispersion is asymptotically proportional to $\sqrt{t}$, as it should be. The approximation (4.9), for which $\bar{\nu} - g = \nu - \mu$, predicts, however, that for $\mu > \nu$ the growth of the dispersion becomes exponential. Results of the Appendix B show that such exponential growth is an artefact of the approximation (4.9). Therefore the approximation (4.19) is—at least in this respect—a distinct improvement over the approximation (4.9).

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[19] This statement requires some explanation. In fact, almost any stochastic flow $\dot{X}(t)$ driven by a colored Markovian noise is a non-Markovian process by itself. In this sense, there is vast literature on the non-Markovian stochastic processes, though this fact is mentioned explicitly very rarely. Also, we omit here the discussion of the so-called exact non-Markovian master equations and exact non-Markovian Langevin equations (for information and literature cf., e.g., Refs. [28,25–27]). What we are saying is that, to the best of the author's knowledge, almost absent from the literature is the use of non-Markovian noises as driving processes.

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